

# Symmetric Coalgebras

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## Abstract

We construct a structure of a ring with local units on a co-Frobenius coalgebra. We study a special class of co-Frobenius coalgebras whose objects we call symmetric coalgebras. We prove that any semiperfect coalgebra can be embedded in a symmetric coalgebra. A dual version of Brauer's equivalence theorem is presented, allowing a characterization of symmetric coalgebras by comparing certain functors. We define an automorphism of the ring with local units constructed from a co-Frobenius coalgebra, which we call the Nakayama automorphism. This is used to give a new characterization to symmetric coalgebras and to describe Hopf algebras that are symmetric as coalgebras. As a corollary we obtain as a consequence the known characterization of Hopf algebras that are symmetric as algebras.

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## 0 Introduction and Preliminaries

Frobenius algebras appeared in group representation theory around 100 years ago. Afterwards they were recognized in many fields of mathematics: commutative algebra, topology, quantum field theory, von Neumann algebras, Hopf algebras, quantum Yang-Baxter equation, etc; see [6], [17] for classical aspects, and [5], [11], [15], [24] for more recent developments. For instance in [15] ideas about Frobenius algebras, Hopf subalgebras, solutions of the Yang-Baxter equation, the Jones polynomial and 2-dimensional topological quantum

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field theories are connected. An important class of algebras, which lies between the class of Frobenius algebras and the class of semisimple algebras, consists of symmetric algebras, as showed by Eilenberg and Nakayama in 1955. Symmetric algebras play a special role in representation theory by the fact that for such a  $k$  algebra  $R$ , the  $k$ -dual functor is naturally equivalent to the  $R$ -dual functor; we refer to the monographs [6] and [17] and the references indicated there. For Hopf algebras, the characterization of finite dimensional Hopf algebras that are symmetric as algebras was given by Oberst and Schneider in [23]. The dual concept of co-Frobenius coalgebra, was initiated by Lin in [20] for not necessarily finite dimensional coalgebras. Several homological characterizations and properties have been evidenced for such coalgebras. A fundamental result, that emphasized the role of this class of coalgebras, is that a Hopf algebra is co-Frobenius as a coalgebra if and only if it has non-zero integrals. This was based on the previous work [18] of Larson and Sweedler on integrals for Hopf algebras. This fact led to a study of Hopf algebras with non-zero integrals from a coalgebraic point of view, which has proved to be very efficient and produced natural easy proofs of facts like the uniqueness of the integrals. The main aim of this paper is to define and study a special class of co-Frobenius coalgebras which we call symmetric coalgebras. In the finite dimensional case, they are precisely duals of symmetric algebras. In the infinite dimensional case some completely new aspects show up. We also study Hopf algebras that are symmetric as coalgebras.

In Section 1 we review some facts about semiperfect and co-Frobenius coalgebras, and we describe the connection to bilinear forms. A characterization for left and right semiperfect coalgebras that are left and right co-Frobenius is given. In Section 2, we define two structures of a ring (without unit) on a co-Frobenius coalgebra, by transferring the structure of a ring with local units from the rational part of the dual algebra. We prove that in fact the two structures that we obtain are the same. The idea to transfer the multiplication in this way already appeared for compact quantum groups in [25], where the convolution product was defined via the inverse of the Fourier transform. The fact was extended in [2] to Hopf algebras with non-zero integrals. A dual construction was studied in [1], where a coalgebra structure was constructed on a Frobenius algebra by transporting back the coalgebra structure of the dual coalgebra. In Section 3 we define the concept of symmetric coalgebra. We give equivalent conditions that define this concept, including one which uses some sort of a trace map with respect to the ring structure of a co-Frobenius coalgebra, and then we present several constructions that produce symmetric coalgebras. In particular we show that cosemisimple coalgebras are symmetric. In Section 4 we prove that any semiperfect coalgebra can be embedded in a symmetric coalgebra by taking a certain trivial coextension. In Section 5 we prove a result dual to Brauer's equivalence theorem, and we give another characterization of symmetric coalgebras by comparing some functors between corepresentations and representations. In Section 6 we define an automorphism of the ring with local units constructed on a co-Frobenius coalgebra, and we call it the Nakayama automorphism. This is used to give a new characterization of symmetric coalgebras, and also in Section 7 to prove that a Hopf algebra  $H$  is symmetric as a coalgebra if and only if it is unimodular and the

dual of the square of the antipode is an inner automorphism with respect to the action of an invertible element of the dual algebra  $H^*$ . As a consequence we obtain the result of Oberst and Schneider that describes Hopf algebras which are symmetric as algebras. We note that we describe explicitly the symmetric bilinear form making  $H$  a symmetric coalgebra, which might be interesting for mathematical physicists. We also derive some consequences concerning the antipode of a cosemisimple Hopf algebra. Finally we discuss some facts about Hopf subalgebras that are symmetric as coalgebras.

We work over a fixed field  $k$ . The category of right comodules over a coalgebra  $C$  is denoted by  $\mathcal{M}^C$ . If  $M$  is a right (left)  $C$ -comodule, we freely regard  $M$  as a left (right)  $C^*$ -module, too. In particular  $C$  is a  $(C^*, C^*)$ -bimodule, and the left (respectively right) action of  $c^* \in C^*$  on  $x \in C$  is denoted by  $c^* \cdot x$  (respectively  $x \cdot c^*$ ). A right  $C$ -comodule  $M$  is called quasi-finite if for any simple right  $C$ -comodule  $S$ , the space  $\text{Hom}_{C^*}(S, M)$  is finite dimensional. If we denote the socle of  $M$  by  $s(M)$ , this is equivalent to the fact that any simple comodule appears with finite multiplicity in  $s(M)$ .

If  $C$  is a coalgebra and  $M$  is a left (or right)  $C^*$ -module, we denote by  $\text{Rat}(M)$  the rational part of  $M$ , which is the largest submodule of  $M$  whose module structure is induced by a right (respectively left)  $C$ -comodule structure. A coalgebra  $C$  is called right semiperfect (see [20]) if the category  $\mathcal{M}^C$  has enough projectives, and this is equivalent to the fact that the injective envelope of any simple left  $C$ -comodule is finite dimensional.  $C$  is right semiperfect if and only if  $\text{Rat}(C^*C^*)$  is dense in  $C^*$  in the finite topology. A coalgebra  $C$  is called left semiperfect if the opposite coalgebra  $C^{\text{cop}}$  is right semiperfect. If  $C$  is left semiperfect and right semiperfect, we simply say that  $C$  is semiperfect. It is known (see [8, Corollary 3.2.17]) that if  $C$  is semiperfect, then  $\text{Rat}(C^*C^*) = \text{Rat}(C^*_{C^*})$ , and we denote this by  $\text{Rat}(C^*)$ . Also  $\text{Rat}(C^*)$  is a ring with local units, i.e. for any finite subset  $X$  of  $\text{Rat}(C^*)$  there exists an idempotent  $e \in \text{Rat}(C^*)$  such that  $ex = xe = x$  for any  $x \in X$ . These idempotent elements (local units) can be defined in terms of the injective envelopes of the simple left (or right)  $C$ -comodules. Most of the coalgebras we work with are semiperfect. However, since we also prove results for arbitrary coalgebras, we prefer to mention each time what sort of a coalgebra we work with. For basic facts and notation about coalgebras and Hopf algebras we refer to [8] and [22].

## 1 Co-Frobenius coalgebras

We first need the following general result.

**Lemma 1.1** *Let  $C$  be an arbitrary coalgebra. Let  $M$  be an injective and quasifinite right  $C$ -comodule, and let  $N$  be a right  $C$ -comodule isomorphic to  $M$ . If  $u : M \rightarrow N$  is an injective morphism of right  $C$ -comodules, then  $u$  is an isomorphism.*

**Proof:** We have that  $s(M) \simeq u(s(M)) \subseteq s(N)$ . Since  $M$  is quasifinite,  $N$  must be quasifinite and  $s(M) \simeq s(N)$ . Thus the finite multiplicity of any simple comodule is the same in  $s(M)$  and  $s(N)$ , showing that  $u(s(M)) = s(N)$ . On the other hand, since  $M$  is injective, there exists a right  $C$ -comodule  $Y$  such that  $N = u(M) \oplus Y$ . If  $Y \neq 0$ , then  $Y$  contains a simple subcomodule, contradicting the fact that  $u(s(M)) = s(N)$ . Thus  $Y = 0$ , and then  $u$  is an isomorphism.  $\blacksquare$

We recall that a coalgebra  $C$  is called left (right) co-Frobenius if there exists a monomorphism of left (right)  $C^*$ -modules from  $C$  to  $C^*$ . If  $C$  is left and right co-Frobenius we say that  $C$  is co-Frobenius. A left (right) co-Frobenius coalgebra is left (right) semiperfect. In particular, for any co-Frobenius coalgebra  $C$  we have that  $\text{Rat}_{(C^*C^*)} = \text{Rat}(C^*)$ , which we denote by  $\text{Rat}(C^*)$ . Note that if  $C$  is left co-Frobenius via  $\alpha : C \rightarrow C^*$ , then in fact  $\alpha$  is a morphism from  $C$  to  $\text{Rat}_{(C^*C^*)}$ . It is proved in [13, Theorem 2.1] that for a co-Frobenius coalgebra we have that  $C$  and  $\text{Rat}(C^*)$  are isomorphic as left  $C^*$ -modules, and also as right  $C^*$ -modules. Since  $C$  is injective and quasifinite as a left (or right)  $C$ -comodule, an immediate consequence of Lemma 1.1 is the following.

**Corollary 1.2** *Let  $C$  be a co-Frobenius coalgebra and  $\alpha : C \rightarrow C^*$  a monomorphism of left (right)  $C^*$ -modules. Then  $\text{Im}(\alpha) = \text{Rat}(C^*)$ , thus  $\alpha$  induces an isomorphism from  $C$  to  $\text{Rat}(C^*)$ .*

We also recall that a bilinear form  $B : C \times C \rightarrow k$  is called:

- *$C^*$ -balanced* if  $B(x \cdot c^*, y) = B(x, c^* \cdot y)$  for any  $x, y \in C, c^* \in C^*$ .
- *left (resp. right) non-degenerate* if  $B(C, y) = 0$  (resp.  $B(y, C) = 0$ ) implies that  $y = 0$ .
- *non-degenerate* if it is left non-degenerate and right non-degenerate.
- *symmetric* if  $B(x, y) = B(y, x)$  for any  $x, y \in C$ .

The following two lemmas are standard results about bilinear forms (see for example [20, Proposition 1] or [9, Lemma 1]). We state them for a semiperfect coalgebra  $C$ .

**Lemma 1.3** *There is a bijective correspondence between the morphisms of left  $C^*$ -modules  $\alpha : C \rightarrow \text{Rat}(C^*)$  and the bilinear forms  $B : C \times C \rightarrow k$  that are  $C^*$ -balanced. The correspondence is described by  $B(x, y) = \alpha(y)(x)$ . Moreover,  $\alpha$  is injective if and only if  $B$  is left non-degenerate.*

**Proof:** We see that  $B(x \cdot c^*, y) = \alpha(y)(x \cdot c^*) = (c^* \alpha(y))(x)$  and  $B(x, c^* \cdot y) = \alpha(c^* \cdot y)(x)$  for any  $x, y \in C, c^* \in C^*$ , showing that  $\alpha$  is left  $C^*$ -linear if and only if  $B$  is  $C^*$ -balanced. The last part follows from the fact that  $\alpha(y) = 0$  if and only if  $B(C, y) = 0$ .  $\blacksquare$

Similarly one obtains the right hand side version of the above result.

**Lemma 1.4** *There is a bijective correspondence between the morphisms of right  $C^*$ -modules  $\beta : C \rightarrow \text{Rat}(C^*)$  and the bilinear forms  $B : C \times C \rightarrow k$  that are  $C^*$ -balanced. The correspondence is described by  $B(x, y) = \beta(x)(y)$ . Moreover,  $\beta$  is injective if and only if  $B$  is right non-degenerate.*

Let now  $C$  be a co-Frobenius coalgebra, and  $\alpha : C \rightarrow C^*$  be an injective morphism of left  $C^*$ -modules. Let  $B : C \times C \rightarrow k$  be the left non-degenerate,  $C^*$ -balanced bilinear form induced by  $\alpha$ , i.e.  $B(x, y) = \alpha(y)(x)$  for any  $x, y \in C$ .

**Lemma 1.5**  *$B$  is also right non-degenerate.*

**Proof:** We have to show that  $B(x, C) = 0$  implies  $x = 0$ . Indeed, we then have  $\alpha(y)(x) = 0$  for any  $y \in C$ . Since  $\text{Im}(\alpha) = \text{Rat}(C^*)$  by Corollary 1.2, and  $\text{Rat}(C^*)$  is dense in  $C^*$ , we see that  $c^*(x) = 0$  for any  $c^* \in C^*$ , showing that  $x = 0$ . ■

Now by Lemma 1.4, the map  $\beta : C \rightarrow C^*$ ,  $\beta(x)(y) = B(x, y) = \alpha(y)(x)$  is an injective morphism of right  $C^*$ -modules, and again by Corollary 1.2 it induces an isomorphism between  $C$  and  $\text{Rat}(C^*)$ .

**Proposition 1.6** *Let  $C$  be a semiperfect coalgebra. Then  $C$  is co-Frobenius if and only if there exists a non-degenerate bilinear form  $B : C \times C \rightarrow k$  which is  $C^*$ -balanced.*

**Proof:** If  $C$  is co-Frobenius, the existence of  $B$  was proved above. For the other way around, if such a form  $B$  exists, then  $C$  is left co-Frobenius by Lemma 1.3 and right co-Frobenius by Lemma 1.4. ■

**Remark 1.7** *There is another way we can construct the morphism of right  $C^*$ -modules  $\beta : C \rightarrow C^*$  from the morphism of left  $C^*$ -modules  $\alpha : C \rightarrow C^*$ . If we regard  $\alpha : C \rightarrow \text{Rat}(C^*)$  as an isomorphism of left  $C^*$ -modules, then it induces an isomorphism of right  $C^*$ -modules between the duals, and hence an isomorphism of right  $C^*$ -modules  $\alpha^* : \text{Rat}(\text{Rat}(C^*)^*) \rightarrow \text{Rat}(C^*)$ . Now since  $C$  is left and right semiperfect, by [12, Theorem 3.5] there exists an isomorphism of right  $C^*$ -modules  $\sigma_C : C \rightarrow \text{Rat}(\text{Rat}(C^*)^*)$  defined by  $\sigma_C(c)(c^*) = c^*(c)$ . By composing these maps, we obtain a monomorphism of right  $C^*$ -modules  $\beta : C \rightarrow \text{Rat}(C^*)$ ,  $\beta = \alpha^* \sigma_C$ . We have that  $\beta(x)(y) = ((\alpha^* \sigma_C)(x))(y) = (\sigma_C(x)\alpha)(y) = \alpha(y)(x)$ , so we obtain exactly the morphism  $\beta$  from above.*

## 2 A ring structure on a co-Frobenius coalgebra

In this section we assume that  $C$  is a co-Frobenius coalgebra. As in the previous section, we denote by  $\alpha : C \rightarrow C^*$  an injective morphism of left  $C^*$ -modules, by  $B : C \times C \rightarrow$

$k, B(x, y) = \alpha(y)(x)$  the associated non-degenerate  $C^*$ -balanced bilinear form, and by  $\beta : C \rightarrow C^*, \beta(x)(y) = B(x, y)$  the associated injective morphism of right  $C^*$ -modules. We can transfer to  $C$  the structure of a ring without identity of  $\text{Rat}(C^*)$  through the inverses of  $\alpha$  and  $\beta$  (in fact through the isomorphisms induced by these), obtaining two multiplications on the space  $C$ , each of them making it a ring with local units. If we denote these multiplications by  $\circ$  and  $\odot$ , then

$$\begin{aligned} x \circ y &= \alpha^{-1}(\alpha(x)\alpha(y)) \\ &= \alpha(x)\alpha^{-1}(\alpha(y)) \quad (\text{since } \alpha \text{ is left } C^* - \text{linear}) \\ &= \alpha(x) \cdot y \\ &= \sum \alpha(x)(y_2)y_1 \end{aligned}$$

and similarly

$$\begin{aligned} x \odot y &= \beta^{-1}(\beta(x)\beta(y)) \\ &= \beta^{-1}(\beta(x)) \cdot \beta(y) \quad (\text{since } \beta \text{ is right } C^* - \text{linear}) \\ &= x \cdot \beta(y) \\ &= \sum \beta(y)(x_1)x_2 \\ &= \sum \alpha(x_1)(y)x_2 \end{aligned}$$

**Proposition 2.1** *For any  $x, y \in C$  we have  $x \circ y = x \odot y$ , thus the two multiplications induced on  $C$  by  $\alpha$  and  $\beta$  coincide.*

**Proof:** Let  $c^* \in C^*$  and  $x, y \in C$ . Then

$$\begin{aligned} c^*(x \circ y) &= \sum \alpha(x)(y_2)c^*(y_1) \\ &= (c^*\alpha(x))(y) \\ &= \alpha(c^* \cdot x)(y) \\ &= B(y, c^* \cdot x) \end{aligned}$$

and

$$\begin{aligned} c^*(x \odot y) &= \sum \alpha(x_1)(y)c^*(x_2) \\ &= \sum \beta(y)(x_1)c^*(x_2) \\ &= (\beta(y)c^*)(x) \\ &= \beta(y \cdot c^*)(x) \\ &= B(y \cdot c^*, x) \end{aligned}$$

Since  $B$  is  $C^*$  balanced we see that  $c^*(x \circ y) = c^*(x \odot y)$  for any  $c^* \in C^*$ , which implies that  $x \circ y = x \odot y$ . ■

**Theorem 2.2** *Let  $C$  be a co-Frobenius coalgebra. Then  $(C, \circ)$  is a ring with local units such that the multiplication is a morphism of  $(C^*, C^*)$ -bimodules.*

**Proof:** It remains to prove that  $((c^* \cdot x) \circ y) = c^* \cdot (x \circ y)$  and  $x \circ (y \cdot c^*) = (x \circ y) \cdot c^*$  for any  $x, y \in C, c^* \in C^*$ . For the first relation we have that

$$\begin{aligned}
(c^* \cdot x) \circ y &= \sum c^*(x_2)x_1 \circ y \\
&= \sum c^*(x_2)\alpha(x_1)(y_2)y_1 \\
&= \sum B(y_2, x_1)c^*(x_2)y_1 \\
&= \sum B(y_2, c^* \cdot x)y_1 \\
&= \sum B(y_2 \cdot c^*, x)y_1 \\
&= \sum B(y_3, x)c^*(y_2)y_1 \\
&= \sum \alpha(x)(y_3)c^*(y_2)y_1 \\
&= c^* \cdot (\sum \alpha(x)(y_2)y_1) \\
&= c^* \cdot (x \circ y)
\end{aligned}$$

while for the second one we see that

$$\begin{aligned}
x \circ (y \cdot c^*) &= \sum c^*(y_1)x \circ y_2 \\
&= \sum c^*(y_1)\alpha(x)(y_3)y_2 \\
&= \sum \alpha(x)(y_2)y_1 \cdot c^* \\
&= (x \circ y) \cdot c^*
\end{aligned}$$

■

We recall that for a ring  $R$  which does not necessarily have identity, a left  $R$ -module  $M$  is called unital if  $RM = M$ . The category of unital left  $R$ -modules is denoted by  $R - uMod$ . The following describes the unital modules for the ring  $(C, \circ)$ .

**Theorem 2.3** *Let  $C$  be a co-Frobenius coalgebra. Then the category of left unital modules  $C - uMod$  associated to the ring with local units  $(C, \circ)$  is isomorphic to the category  $\mathcal{M}^C$  of right comodules over the coalgebra  $C$ .*

**Proof:** Since the rings  $(C, \circ)$  and  $Rat(C^*)$  are isomorphic, the categories of unital left modules  $C - uMod$  and  $Rat(C^*) - uMod$  are isomorphic. But for a semiperfect coalgebra  $Rat(C^*) - uMod$  is isomorphic to  $\mathcal{M}^C$  by [4, Proposition 2.7]. ■

The category isomorphism from the theorem induces immediately the following.

**Corollary 2.4** *Let  $M$  be a right  $C$ -comodule. Then the lattices of subobjects of the right  $C$ -comodule  $M$  and of the unital left  $C$ -module  $M$  are isomorphic.*

**Corollary 2.5** *Let  $I$  be a subspace of the co-Frobenius coalgebra  $C$ . Then  $I$  is a left (right) ideal of the ring  $(C, \circ)$  if and only if  $I$  is a right (left) coideal of the coalgebra  $C$ .*

### 3 Symmetric coalgebras

A finite dimensional algebra  $A$  over the field  $k$  is called symmetric if there exists a symmetric non-degenerate bilinear form  $\langle \cdot, \cdot \rangle : A \times A \rightarrow k$  such that  $\langle ab, c \rangle = \langle a, bc \rangle$  for any  $a, b, c \in A$ . We have the following characterization of symmetric algebras.

**Theorem 3.1** ([17, Theorem 16.54]) *Let  $A$  be a finite dimensional algebra. The following assertions are equivalent.*

- (1)  *$A$  is symmetric.*
- (2)  *$A$  and  $A^*$  are isomorphic as  $(A, A)$ -bimodules.*
- (3) *There exists a  $k$ -linear map  $f : A \rightarrow k$  such that  $f(xy) = f(yx)$  for any  $x, y \in A$ , and  $\text{Ker}(f)$  does not contain a non-zero left ideal.*

The aim of this section is to define the dual property for coalgebras. Note that we do not restrict to finite dimensional coalgebras.

**Definition 3.2** *A coalgebra  $C$  is called symmetric if there exists an injective morphism  $\alpha : C \rightarrow C^*$  of  $(C^*, C^*)$ -bimodules.*

Obviously, a symmetric coalgebra is co-Frobenius. Also, for a finite dimensional coalgebra  $C$  we have that  $C$  is symmetric if and only if the dual algebra  $C^*$  is symmetric. Symmetric coalgebras can be defined in an alternative way, as the next result shows.

**Theorem 3.3** *Let  $C$  be a coalgebra. The following assertions are equivalent.*

- (1)  *$C$  is a symmetric coalgebra.*
- (2) *There exists a bilinear form  $B : C \times C \rightarrow k$  which is symmetric, non-degenerate and  $C^*$ -balanced.*
- (3)  *$C$  is co-Frobenius, and there exists a linear map  $f : C \rightarrow k$  such that denoting by  $(C, \circ)$  the ring with local units defined as in Theorem 2.2 we have*
  - i)  $f(x \circ y) = f(y \circ x)$  for any  $x, y \in C$ .
  - ii)  $f(c^* \cdot x) = f(x \cdot c^*)$  for any  $x \in C, c^* \in C^*$ .
  - iii)  $\text{Ker}(f)$  does not contain a non-zero left (or right) co-ideal of  $C$ .



**Proof:** (1)  $\Rightarrow$  (2) Let  $\alpha : C \rightarrow C^*$  be an injective morphism of  $(C^*, C^*)$ -bimodules. Define  $B : C \times C \rightarrow k$  by  $B(x, y) = \alpha(y)(x)$  for any  $x, y \in C$ . Then by Lemma 1.3,  $B$  is bilinear,  $C^*$ -balanced and left non-degenerate. Let us consider the multiplication  $\circ$  defined on  $C$  by  $x \circ y = \alpha^{-1}(\alpha(x)\alpha(y)) = \alpha(x) \cdot y = x \cdot \alpha(y)$ . Then

$$B(x \circ z, y) = B(x \cdot \alpha(z), y) = B(x, \alpha(z) \cdot y) = B(x, z \circ y)$$

Note that for any  $x, y \in C, c^* \in C^*$  we have

$$\begin{aligned} B(c^* \cdot x, y) &= \alpha(y)(c^* \cdot x) \\ &= (\alpha(y)c^*)(x) \\ &= \alpha(y \cdot c^*)(x) \\ &= B(x, y \cdot c^*) \end{aligned}$$

Hence

$$\begin{aligned} B(z \circ x, y) &= B(\alpha(z) \cdot x, y) \\ &= B(x, y \cdot \alpha(z)) \\ &= B(x, y \circ z) \end{aligned}$$

Let  $x, y \in C$ . Since  $C$  has local units, there exists  $e \in C$  with  $e \circ x = x \circ e = x$  and  $e \circ y = y \circ e = y$ . Then

$$\begin{aligned} B(x, y) &= B(x, y \circ e) \\ &= B(x \circ y, e) \\ &= B(y, e \circ x) \\ &= B(y, x) \end{aligned}$$

so  $B$  is symmetric. This shows that  $B$  is also right non-degenerate.

(2)  $\Rightarrow$  (1) Define  $\alpha : C \rightarrow C^*$  by  $\alpha(y)(x) = B(x, y)$ . Then  $\alpha$  is an injective morphism of  $C^*, C^*$ -bimodules by Lemmas 1.3 and 1.4.

(2)  $\Rightarrow$  (3) Clearly  $C$  is co-Frobenius. Let  $\alpha : C \rightarrow C^*$  be an injective morphism of  $(C^*, C^*)$ -bimodules and let  $\circ$  be the multiplication induced on  $C$ . Define  $f : C \rightarrow k$  as follows. Let  $x \in C$ . Then there exists an idempotent  $e \in C$  such that  $x \circ e = e \circ x = x$ . We set  $f(x) = B(x, e)$ .

We first show that  $f$  is well defined. Indeed, if  $e' \in C$  is another idempotent with  $e' \circ x = x \circ e' = x$ , then

$$\begin{aligned} B(x, e') &= B(e \circ x, e') \\ &= B(x, e' \circ e) \\ &= B(x \circ e', e) \\ &= B(x, e) \end{aligned}$$

We have used the relation already proved in (1)  $\Rightarrow$  (2).

Let  $x, y \in C$ , and pick an idempotent  $e \in C$  such that  $x \circ e = e \circ x = x$  and  $y \circ e = e \circ y = y$ . Then we have

$$\begin{aligned}
f(x \circ y) &= B(x \circ y, e) \\
&= B(y, e \circ x) \\
&= B(y, x) \\
&= B(y, x \circ e) \\
&= B(y \circ x, e) \\
&= f(y \circ x)
\end{aligned}$$

which proves (i).

Let now  $x \in C$  and  $c^* \in C^*$ . We show that  $f(c^* \cdot x) = f(x \cdot c^*)$ . Pick an idempotent  $e \in C$  such that  $x \circ e = e \circ x = x$ ,  $(c^* \cdot x) \circ e = e \circ (c^* \cdot x) = c^* \cdot x$  and  $(x \cdot c^*) \circ e = e \circ (x \cdot c^*) = x \cdot c^*$ . Then we have

$$\begin{aligned}
f(x \cdot c^*) &= B(x \cdot c^*, e) \\
&= B((x \circ e) \cdot c^*, e) \\
&= B(x \circ (e \cdot c^*), e) \\
&= B(e \cdot c^*, e \circ x) \\
&= B(e \cdot c^*, x) \\
&= B(e, c^* \cdot x) \\
&= B(c^* \cdot x, e) \\
&= f(c^* \cdot x)
\end{aligned}$$

Finally, to show (iii), assume that  $I \subseteq \text{Ker}(f)$  for a right coideal  $I$  of  $C$ . Then by Corollary 2.5 we have that  $I$  is a left ideal in the ring  $(C, \circ)$ . Let  $x \in I$ . For any  $c \in C$  pick an idempotent  $e_c \in C$  such that  $x \circ e_c = e_c \circ x = x$  and  $c \circ e_c = e_c \circ c = c$ . Then

$$\begin{aligned}
B(c, x) &= B(c, x \circ e_c) \\
&= B(c \circ x, e_c) \\
&= f(c \circ x) \\
&= 0
\end{aligned}$$

which implies that  $x = 0$  by the non-degeneracy of  $B$ . Thus  $I = 0$ .

(3)  $\Rightarrow$  (2) Define  $B : C \times C \rightarrow k$  by  $B(x, y) = f(x \circ y)$ . Then clearly  $B$  is bilinear and symmetric. If  $B(C, x) = 0$ , then  $f(C \circ x) = 0$ , so the left coideal  $C \circ x$  must be zero. Since  $C$  has local units this implies that  $x = 0$ . Thus  $B$  is non-degenerate. We finally show that  $B$  is  $C^*$ -balanced. Indeed, for  $x, y \in C$  and  $c^* \in C^*$  we have that

$$B(x \cdot c^*, y) = f((x \cdot c^*) \circ y)$$

$$\begin{aligned}
&= f(y \circ (x \cdot c^*)) \\
&= f((y \circ x) \cdot c^*) \\
&= f(c^* \cdot (y \circ x)) \\
&= f((c^* \cdot y) \circ x) \\
&= f(x \circ (c^* \cdot y)) \\
&= B(x, c^* \cdot y)
\end{aligned}$$

and this ends the proof. ■

**Remark 3.4** *We note that if  $C$  is a symmetric coalgebra, then  $(x \cdot c^*) \circ y = x \circ (c^* \cdot y)$  for any  $x, y \in C$ ,  $c^* \in C^*$ . Indeed we have that  $(\alpha(x)c^*)\alpha(y) = \alpha(x)(c^*\alpha(y))$ . Since  $\alpha$  is a morphism of left and right  $C^*$ -modules, we obtain  $\alpha(x \cdot c^*)\alpha(y) = \alpha(x)\alpha(c^* \cdot y)$ . The desired relation follows now by applying  $\alpha^{-1}$ , which is an algebra morphism.*

**Proposition 3.5** *Let  $(C_i)_{i \in I}$  be a family of symmetric coalgebras. Then  $C = \oplus_{i \in I} C_i$  is a symmetric coalgebra.*

**Proof:** For any  $i \in I$ , let  $B_i : C_i \times C_i \rightarrow k$  be a symmetric, non-degenerate and  $C_i^*$ -balanced bilinear form. We define the bilinear form  $B : C \times C \rightarrow k$  such that the restriction of  $B$  to  $C_i \times C_i$  is  $B_i$  for any  $i \in I$ , and  $B(x, y) = 0$  for any  $x \in C_i, y \in C_j$  with  $i \neq j$ . Then clearly  $B$  is symmetric, non-degenerate and  $C^*$ -balanced. ■

**Corollary 3.6** *A cosemisimple coalgebra is symmetric.*

**Proof:** A cosemisimple coalgebra is a direct sum of simple coalgebras. But a simple subcoalgebra is necessarily finite dimensional, and then it is symmetric since its dual, a matrix algebra over a division ring, is a symmetric algebra (see [17, Example 16.59]). ■

**Remark 3.7** *We note that Proposition 3.5 provides in particular an example of an infinite dimensional symmetric coalgebra which is not cosemisimple.*

**Proposition 3.8** *Let  $C$  and  $D$  be symmetric coalgebras. Then  $C \otimes D$  is symmetric.*

**Proof:** Let  $\alpha : C \rightarrow C^*$  be an injective morphism of  $(C^*, C^*)$ -bimodules, and let  $\beta : D \rightarrow D^*$  be an injective morphism of  $(D^*, D^*)$ -bimodules. Then it is straightforward to check that

$\alpha \otimes \beta : C \otimes D \rightarrow C^* \otimes D^* \subseteq (C \otimes D)^*$  is an injective morphism of  $(C^* \otimes D^*, C^* \otimes D^*)$ -bimodules. Since  $Im(\alpha \otimes \beta) \subseteq Rat(C^*) \otimes Rat(D^*)$ , and  $Rat(C^*) \otimes Rat(D^*)$  is dense in  $(C \otimes D)^*$ , we have that  $\alpha \otimes \beta$  is additionally a morphism of  $((C \otimes D)^*, (C \otimes D)^*)$ -bimodules. ■

**Corollary 3.9** *Let  $C$  be a symmetric coalgebra and  $n$  a positive integer. Then the comatrix coalgebra  $M^c(n, C)$  (see [7, Section 3]) is symmetric.*

**Proof:** It follows by Proposition 3.8, Corollary 3.6 and the fact that  $M^c(n, C) \simeq C \otimes M^c(n, k)$ . ■

We note that there exists co-Frobenius coalgebras that are not symmetric. This can be seen by taking an example of a Frobenius (finite dimensional) algebra that is not a symmetric algebra (see for example [17, Example 16.66]) and consider the dual coalgebra. However, in the cocommutative case we have the following.

**Proposition 3.10** *Let  $C$  be a cocommutative coalgebra. Then  $C$  is symmetric if and only if  $C$  is co-Frobenius.*

**Proof:** If  $\alpha : C \rightarrow C^*$  is an injective morphism of left  $C^*$ -modules, then  $\alpha$  is also a morphism of right  $C^*$ -modules, since  $C^*$  is commutative. ■

In the finite dimensional case, we have some more characterizations of symmetric coalgebras.

**Proposition 3.11** *Let  $C$  be a finite dimensional coalgebra. The following assertions are equivalent.*

- (i)  $C$  is symmetric.
- (ii) There exists a cocommutative element  $c \in C$  (i.e.  $\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1$ ) which does not belong to any proper left coideal of  $C$ .
- (iii) The right (or left)  $C^*$ -module  $C$  is cyclic, generated by a cocommutative element.

**Proof:** This follows by a direct dualization of Theorem 3.1, so we just sketch the proof. We have noticed that  $C$  is a symmetric coalgebra if and only if  $C^*$  is a symmetric algebra. If  $f : C^* \rightarrow k$  is a linear map such that  $f(uv) = f(vu)$  for any  $u, v \in C^*$  and  $Ker(f)$  not containing a non-zero left ideal, then let  $c \in C$  such that  $f = i(c)$ , where  $i : C \rightarrow C^{**}$  is the natural linear isomorphism. It is easy to see that  $c$  is a cocommutative element. Since  $Ker(f)$  does not contain a non-zero left ideal, we have that  $(Ker(f))^\perp = \{x \in C | u(x) =$

0 for any  $u \in \text{Ker}(f)\}$  is not contained in a proper left coideal. But  $\text{Ker}(f) = \text{Ker}(i(c)) = c^\perp = \{u \in C^* | u(c) = 0\}$ , so then  $(\text{Ker}(f))^\perp = kc$ , and we obtain the characterization (ii).

To obtain (iii), we find the cocommutative element  $c$  as for (ii), and we note that if  $f(C^*u) = 0$  for some  $u \in C^*$ , we must have  $u = 0$ . Thus if  $\sum c^*(c_1)u(c_2) = 0$  for any  $c^* \in C^*$ , then  $u = 0$ . Therefore if  $u \cdot c = 0$ , then  $u = 0$ . If we take a representation of  $\Delta(c)$  such that the  $c_1$ 's are linearly independent, this implies that if  $u(c_2) = 0$  for any  $c_2$  (in the certain representation of  $\Delta(c)$ ), then  $u = 0$ . But this means that the  $c_2$ 's span  $C$ , i.e.  $c \cdot C^* = C$ .

The converses (ii)  $\Rightarrow$  (i) and (iii)  $\Rightarrow$  (i) follow by reversing the above arguments.  $\blacksquare$

## 4 An embedding theorem

A result of Tachikawa says that any finite dimensional algebra is isomorphic to a quotient of a symmetric algebra (see the book of Lam [17, page 443]). In this section we prove a dual result about coalgebras. We do not restrict to finite dimensional coalgebras. Let  $C$  be an arbitrary coalgebra and let  $M$  be a  $(C, C)$ -bicomodule, i.e.  $M$  is a left  $C$ -comodule with comodule structure given by  $m \mapsto \sum m_{(-1)} \otimes m_{(0)}$ , a right  $C$ -comodule with structure given by  $m \mapsto \sum m_{[0]} \otimes m_{[1]}$ , and

$$\sum m_{(-1)} \otimes (m_{(0)})_{[0]} \otimes (m_{(0)})_{[1]} = \sum (m_{[0]})_{(-1)} \otimes (m_{[0]})_{(0)} \otimes m_{[1]}$$

for any  $m \in M$ . We define a comultiplication and a counit on the space  $D = C \oplus M$  by

$$\begin{aligned} \Delta(c, m) &= \sum (c_1, 0) \otimes (c_2, 0) + \sum (m_{(-1)}, 0) \otimes (0, m_{(0)}) + \sum (0, m_{[0]}) \otimes (m_{[1]}, 0) \\ \varepsilon(c, m) &= \varepsilon(c) \end{aligned}$$

for any  $c \in C, m \in M$ . These make  $D$  into a coalgebra which we call the trivial coextension of  $C$  and  $M$ . Clearly  $C$  is isomorphic to a subcoalgebra of  $D$ . The definition of the trivial coextension is close to the definition of the coalgebra associated to a Morita-Takeuchi context in [7].

Since  $M$  is a  $(C^*, C^*)$ -bimodule, the dual space  $M^*$  has an induced structure of a  $(C^*, C^*)$ -bimodule, therefore we can consider the trivial extension  $C^* \oplus M^*$ , which has an algebra structure with the multiplication given by  $(c^*, m^*)(b^*, n^*) = (c^*b^*, c^*n^* + m^*b^*)$  for any  $c^*, b^* \in C^*, m^*, n^* \in M^*$ . The next result follows by a straightforward computation.

**Proposition 4.1** *The dual algebra of  $D = C \oplus M$  is isomorphic to the trivial extension  $C^* \oplus M^*$ .*

Let us take now a semiperfect coalgebra  $C$  and  $M = \text{Rat}(C^*)$  with the natural structure of a  $(C, C)$ -bicomodule induced by the structures of a rational  $(C^*, C^*)$ -bimodule. Then we can form the trivial coextension  $C \oplus M = C \oplus \text{Rat}(C^*)$ .

**Theorem 4.2** *Let  $C$  be a semiperfect coalgebra. Then the trivial coextension  $C \oplus \text{Rat}(C^*)$  is a symmetric coalgebra. In particular any semiperfect coalgebra can be embedded in a symmetric coalgebra.*

**Proof:** Denote  $M = \text{Rat}(C^*)$ ,  $D = C \oplus M = C \oplus \text{Rat}(C^*)$ , and identify  $D^*$  with  $C^* \oplus M^*$ . Since  $\text{Rat}(C^*)$  is dense in  $C^*$ , the morphism of  $C^*, C^*$ -bimodules  $\sigma : C \rightarrow M^*$  defined by  $\sigma(c)(m) = m(c)$  for any  $c \in C, m \in M$ , is injective.

Now define the map  $\alpha : D \rightarrow D^*$  by  $\alpha(c, m) = (m, \sigma(c))$  for any  $c \in C, m \in M$ . We first show that  $\alpha$  is a morphism of left  $D^*$ -modules. First note that since  $\sigma$  is left  $C^*$ -linear we have that

$$c^* \sigma(c) = \sigma(c^* \cdot c) = \sum \sigma(c^*(c_2)c_1) = \sum c^*(c_2)\sigma(c_1)$$

for any  $c \in C, c^* \in C^*$ . Also for any  $n^* \in M^*, m \in M$  we have  $\sum n^*(m_{(0)})\sigma(m_{(-1)}) = n^*m$ . Indeed, for any  $b^* \in M$  we have

$$\begin{aligned} \sum (n^*(m_{(0)})\sigma(m_{(-1)}))(b^*) &= \sum n^*(m_{(0)})b^*(m_{(-1)}) \\ &= n^*(mb^*) \\ &= (n^*m)(b^*) \end{aligned}$$

Now for any  $(c^*, n^*) \in D^*, (c, m) \in D$ , we have that

$$\begin{aligned} \alpha((c^*, n^*)(c, m)) &= \alpha(\sum c^*(c_2)(c_1, 0) + \sum n^*(m_{(0)})(m_{(-1)}, 0) + \sum c^*(m_{[1]})(0, m_{[0]})) \\ &= \sum c^*(c_2)(0, \sigma(c_1)) + \sum n^*(m_{(0)})(0, \sigma(m_{(-1)})) + \sum c^*(m_{[1]})(m_{[0]}, 0) \\ &= (0, c^*\sigma(c)) + (0, n^*m) + (c^*m, 0) \\ &= (c^*m, n^*m + c^*\sigma(c)) \\ &= (c^*, n^*)(m, \sigma(c)) \end{aligned}$$

Similarly we show that  $\alpha$  is right  $D^*$ -linear. Since  $\sigma$  is right  $C^*$ -linear we have as above  $\sum c^*(c_1)\sigma(c_2) = \sigma(c)c^*$  for any  $C \in C, c^* \in C^*$ . We also see by a direct computation that  $mn^* = \sum n^*(m_{[0]})\sigma(m_{[1]})$ . Hence

$$\begin{aligned} \alpha((c, m)(c^*, n^*)) &= \alpha(\sum c^*(c_1)(c_2, 0) + \sum c^*(m_{(-1)})(0, m_{(0)}) + \sum n^*(m_{[0]})(m_{[1]}, 0)) \\ &= \sum c^*(c_1)(0, \sigma(c_2)) + \sum c^*(m_{(-1)})(m_{(0)}, 0) + \sum n^*(m_{[0]})(0, \sigma(m_{[1]})) \\ &= (0, \sigma(c)c^*) + (mc^*, 0) + (0, mn^*) \\ &= (mc^*, mn^* + \sigma(c)c^*) \\ &= (m, \sigma(c))(c^*, n^*) \end{aligned}$$

Clearly  $\alpha$  is injective, and we conclude that  $C$  can be embedded in the symmetric coalgebra  $D$ . ■

**Remark 4.3** We note that any subcoalgebra of a symmetric coalgebra is semiperfect. This follows by the fact that a subcoalgebra of a left (right) semiperfect coalgebra is also left (right) semiperfect (see [8, Corollary 3.2.11]). Thus the condition that  $C$  is semiperfect is necessary for embedding  $C$  in a symmetric coalgebra.

## 5 A coalgebra version for the Brauer Equivalence Theorem

Let  $C$  be an arbitrary coalgebra. We consider the contravariant functors

$$\begin{aligned} F : \mathcal{M}^C &\longrightarrow \text{mod} - C^*, \quad F(M) = \text{Hom}_k({}_{C^*}M, k) \\ G : \mathcal{M}^C &\longrightarrow \text{mod} - C^*, \quad G(M) = \text{Hom}_{C^*}({}_{C^*}M, {}_{C^*}C_{C^*}) \\ H : \mathcal{M}^C &\longrightarrow \text{mod} - C^*, \quad H(M) = \text{Hom}_{C^*}({}_{C^*}M, C_{C^*}^*) \end{aligned}$$

where  $\text{mod} - C^*$  denotes the category of right  $C^*$ -modules. There is no danger of confusion if we denote by  $\leftarrow$  the right action of  $C^*$  on any of  $F(M), G(M), H(M)$ . The following is a coalgebra version for the Brauer equivalence theorem.

**Theorem 5.1** *The functors  $F$  and  $G$  are naturally equivalent.*

**Proof:** For any  $M \in \mathcal{M}^C$  we define  $\alpha(M) : G(M) \rightarrow F(M)$  by  $\alpha(M)(f) = \varepsilon f$  for any  $f \in G(M)$ . We see that  $\alpha(M)$  is a morphism of right  $C^*$ -modules since

$$\begin{aligned} (\alpha(M)(f \leftarrow c^*))(m) &= (\varepsilon(f \leftarrow c^*))(m) \\ &= \varepsilon(f(m) \cdot c^*) \\ &= \sum c^*(f(m)_2) \varepsilon(f(m)_1) \\ &= \sum c^*(m_1) \varepsilon(f(m_0)) \\ &= (\varepsilon f)(c^* \cdot m) \\ &= (\alpha(M)(f) \leftarrow c^*)(m) \end{aligned}$$

We also define  $\beta(M) : F(M) \rightarrow G(M)$  by  $(\beta(M)(g))(m) = \sum g(m_0)m_1$  for any  $g \in F(M)$ . We have that  $\beta(M)(g)$  is left  $C^*$ -linear since

$$\begin{aligned} (\beta(M)(g))(c^* \cdot m) &= \sum g((c^* \cdot m)_0)(c^* \cdot m)_1 \\ &= \sum c^*(m_2)g(m_0)m_1 \\ &= c^* \cdot (\beta(M)(g))(m) \end{aligned}$$

Now we have

$$\begin{aligned}
((\beta(M)\alpha(M))(f))(m) &= \sum \varepsilon(f(m_0))m_1 \\
&= \sum \varepsilon(f(m)_1)f(m)_2 \\
&= f(m)
\end{aligned}$$

and

$$\begin{aligned}
((\alpha(M)\beta(M))(g))(m) &= (\varepsilon\beta(M)(g))(m) \\
&= \sum \varepsilon(m_1)g(m_0) \\
&= g(m)
\end{aligned}$$

showing that  $\alpha(M)$  and  $\beta(M)$  are inverse each other. It is also easy to see that  $\alpha$  defines a functorial morphism.  $\blacksquare$

As a biproduct of the above proof, we describe the automorphisms of the right  $C$ -comodule  $C$ . This will be used in the next section. We denote by  $U(A)$  the set of invertible elements of an algebra  $A$ .

**Proposition 5.2** *Let  $C$  be a coalgebra. Then a map  $f : C \rightarrow C$  is an isomorphism of left  $C^*$ -modules if and only if there exists  $u \in U(C^*)$  such that  $f(c) = c \cdot u$  for any  $c \in C$ .*

**Proof:** Let us consider the isomorphism (of right  $C^*$ -modules)  $\beta(C) : C^* \rightarrow \text{Hom}_{C^*}(C, C)$  from the proof of Theorem 5.1. We have that  $(\beta(C)(c^*))(c) = \sum c^*(c_1)c_2 = c \cdot c^*$  for any  $c \in C, c^* \in C^*$ . Then clearly  $\beta(C)(c^*d^*) = \beta(C)(d^*)\beta(C)(c^*)$ , so  $\beta(C)$  is an anti-isomorphism of algebras, and then the result follows by taking the induced bijective correspondence between  $U(C^*)$  and  $U(\text{Hom}_{C^*}(C, C))$ .  $\blacksquare$

Now we can characterize symmetric coalgebras by using the above functors.

**Theorem 5.3** *If  $C$  is a symmetric coalgebra, then the functors  $G$  and  $H$  are naturally equivalent. Conversely, if  $C$  is a semiperfect coalgebra and  $G \simeq H$ , then  $C$  is a symmetric coalgebra.*

**Proof:** Assume that  $C$  is symmetric. Then  $C$  is semiperfect and  $C \simeq \text{Rat}(C^*)$  as  $(C^*, C^*)$ -bimodules. Then for any  $M \in \mathcal{M}^C$  we have

$$H(M) = \text{Hom}_{C^*}(C^*M, C^*C_{C^*}) = \text{Hom}_{C^*}(C^*M, \text{Rat}(C^*)) \simeq \text{Hom}_{C^*}(C^*M, C^*C_{C^*}) = G(M)$$

Conversely, assume that  $C$  is semiperfect and  $G \simeq H$ . Since for any  $M \in \mathcal{M}^C$  we have  $H(M) = \text{Hom}_{C^*}(C^*M, \text{Rat}(C^*))$ , we see that there exists an isomorphism

$$\alpha : \text{Hom}_{C^*}(-, C^*C_{C^*}) \rightarrow \text{Hom}_{C^*}(-, \text{Rat}(C^*))$$



Define the morphism of left  $C^*$ -modules  $u : C \rightarrow \text{Rat}(C^*)$  by  $u = \alpha(C)(1_C)$ , and  $v : \text{Rat}(C^*) \rightarrow C$  by  $v = \alpha^{-1}(\text{Rat}(C^*))(1_{\text{Rat}(C^*)})$ . A standard argument using Yoneda's Lemma shows that  $v$  is the inverse of  $u$ .

We show that  $u$  is a morphism of right  $C^*$ -modules. Let  $c^* \in C^*$ , and consider the map  $f : C \rightarrow C$ ,  $f(x) = x \cdot c^*$ , which is left  $C^*$ -linear. If we apply

$$\text{Hom}(f, 1_C) \circ \alpha(C) = \alpha(C) \circ \text{Hom}(f, 1_C)$$

to  $1_C$ , we obtain that  $uf = \alpha(C)(f)$ . We have that  $(uf)(x) = u(x \cdot c^*)$ . On the other hand

$$(1_C \leftarrow c^*)(x) = 1_C(x) \cdot c^* = x \cdot c^* = f(x)$$

so  $1_C \leftarrow c^* = f$ , and then

$$\begin{aligned} (\alpha(C)(f))(x) &= \alpha(C)((1_C \leftarrow c^*)(x)) \\ &= (\alpha(C)(1_C) \leftarrow c^*)(x) \\ &= ((\alpha(C)(1_C))(x))c^* \\ &= u(x)c^* \end{aligned}$$

Therefore  $u(x \cdot c^*) = u(x)c^*$ , which shows that  $u$  is a morphism of  $(C^*, C^*)$ -bimodules. ■

## 6 The Nakayama automorphism

Let  $C$  be a co-Frobenius coalgebra and  $\circ$  the multiplication defined on  $C$  as in Section 2. Let  $B : C \times C \rightarrow k$  be a non degenerate  $C^*$ -balanced bilinear form, and let

$$\alpha : {}_{C^*}C \rightarrow {}_{C^*}C^*, \quad \alpha(y)(x) = B(x, y)$$

and

$$\beta : C_{C^*} \rightarrow C_{C^*}^*, \quad \beta(y)(x) = B(y, x)$$

be the injective maps associated to  $B$  as in Section 1.

Let  $x \in C$ . Since  $\alpha(x) \in \text{Rat}(C^*C^*) = \text{Rat}(C_{C^*}^*)$ , there exists a unique  $\sigma(x) \in C$  such that  $\alpha(x) = \beta(\sigma(x))$ , i.e.  $B(x, y) = B(\sigma(y), x)$  for any  $y \in C$ . Similarly,  $\beta(x) = \alpha(\tau(x))$  for some  $\tau(x) \in C$ . Then  $\alpha(x) = \beta(\sigma(x)) = \alpha((\tau \circ \sigma)(x))$ , so  $\tau \circ \sigma = 1_C$ . Similarly  $\sigma \circ \tau = 1_C$ , hence  $\sigma$  is bijective. Now, if  $x, y \in C$ , we have

$$\begin{aligned} \beta(\sigma(x \circ y)) &= \alpha(x \circ y) \\ &= \alpha(x)\alpha(y) \\ &= \beta(\sigma(x))\beta(\sigma(y)) \\ &= \beta(\sigma(x) \circ \sigma(y)) \end{aligned}$$

so  $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$ . Therefore  $\sigma : C \rightarrow C$  is an automorphism of the ring  $(C, \circ)$ . We call  $\sigma$  the *Nakayama automorphism* of  $C$ . Note that  $\sigma$  depends on the choice of the bilinear form  $B$ . However, the Nakayama automorphism is determined up to the inner action of an invertible element of  $C^*$ , as the following result shows.

**Proposition 6.1** *Let  $C$  be a co-Frobenius coalgebra,  $B, B' : C \times C \rightarrow k$  be two non-degenerate  $C^*$ -balanced bilinear forms, and  $\sigma, \sigma'$  the associated Nakayama automorphisms. Then there exists  $u \in U(C^*)$  such that  $\sigma'(y) = \sigma(u^{-1} \cdot y \cdot u)$  for any  $y \in C$ .*

**Proof:** Let  $\alpha' : {}_{C^*}C \rightarrow {}_{C^*}C^*$ ,  $\alpha'(y)(x) = B'(x, y)$ . By Proposition 5.2 there exists  $u \in U(C^*)$  such that  $(\alpha^{-1} \circ \alpha')(x) = x \cdot u$ , for any  $x \in C$ . Then  $\alpha'(x) = \alpha(x \cdot u)$  and

$$\begin{aligned} B'(x, y) &= \alpha'(y)(x) \\ &= \alpha(y \cdot u)(x) \\ &= B(x, y \cdot u) \end{aligned}$$

Hence we have that

$$\begin{aligned} B(\sigma'(y), x) &= B(\sigma'(y), x \cdot u^{-1}u) \\ &= B'(\sigma'(y), x \cdot u^{-1}) \\ &= B'(x \cdot u^{-1}, y) \\ &= B(x \cdot u^{-1}, y \cdot u) \\ &= B(x, u^{-1} \cdot y \cdot u) \\ &= B(\sigma(u^{-1} \cdot y \cdot u), x) \end{aligned}$$

and since  $B$  is non-degenerate we must have  $\sigma'(y) = \sigma(u^{-1} \cdot y \cdot u)$ . ■

**Proposition 6.2** *Let  $C$  be a co-Frobenius coalgebra with non-degenerate  $C^*$ -balanced bilinear form  $B : C \times C \rightarrow k$ , and let  $\sigma$  be the associated Nakayama automorphism. Then  $C$  is symmetric if and only if there exists  $u \in U(C^*)$  such that  $\sigma(x) = u^{-1} \cdot x \cdot u$  for any  $x \in C$ .*

**Proof:** Assume that the Nakayama automorphism is interior, so  $B(x, y) = B(y, u^{-1}xu)$  for any  $x, y \in C$ . We define the bilinear map  $B' : C \times C \rightarrow k$ , by  $B'(x, y) = B(u^{-1} \cdot x, y)$ . Then

$$\begin{aligned} B'(y, x) &= B(u^{-1} \cdot y, x) \\ &= B(u^{-1} \cdot x \cdot u, u^{-1} \cdot y) \\ &= B(u^{-1} \cdot x, y) \\ &= B'(x, y) \end{aligned}$$

so  $B'$  is symmetric.  $B'$  is clearly  $C^*$ -balanced since so is  $B$ .

For the converse, assume that  $C$  is symmetric, and let  $B'$  be a symmetric non-degenerate  $C^*$ -balanced bilinear form. The Nakayama automorphism associated to  $B'$  is the identity, and the result follows by applying Proposition 6.1 to  $B$  and  $B'$ .  $\blacksquare$

## 7 Hopf algebras that are symmetric coalgebras

We recall that a left (resp. right) integral on a Hopf algebra  $H$  is an element  $t \in H^*$  such that  $h^*t = h^*(1)t$  (resp.  $th^* = h^*(1)t$ ) for any  $h^* \in H^*$ . A Hopf algebra is co-Frobenius as a coalgebra if and only if it has non-zero left (or right) integrals. In this case the dimension of the space of left (resp. right) integrals is 1, and  $H$  is called unimodular if these two spaces of integrals are equal. Now we are able to describe Hopf algebras whose underlying coalgebra structure is symmetric.

**Theorem 7.1** *Let  $H$  be a Hopf algebra with antipode  $S$ . Then  $H$  is symmetric as a coalgebra if and only if  $H$  is unimodular and there exists  $u \in U(H^*)$  such that  $S^2(h) = u^{-1} \cdot h \cdot u = \sum u(h_1)u^{-1}(h_3)h_2$  for any  $h \in H$ .*

**Proof:** Assume that  $H$  is symmetric. Let  $\alpha : H \rightarrow \text{Rat}(H^*)$  be an isomorphism of  $(H^*, H^*)$ -bimodules, and let  $B : H \times H \rightarrow k$ ,  $B(x, y) = \alpha(y)(x)$ , be the associated bilinear form, which is symmetric, non-degenerate and  $H^*$ -balanced.

Since  $\alpha$  is a morphism of left  $H^*$ -modules, we have that  $h^*\alpha(1) = \alpha(h^* \cdot 1) = \alpha(h^*(1)1) = h^*(1)\alpha(1)$  for any  $h^* \in H^*$ , so  $\alpha(1)$  is a left integral on  $H$ . Similarly, since  $\alpha$  is a morphism of right  $H^*$ -modules, we have that  $\alpha(1)$  is also a right integral on  $H$ , thus  $H$  is unimodular.

Let  $t$  be a non-zero left and right integral on  $H$ . We have by [8, Proposition 5.5.4] that  $tS = t$ . Since  $H$  is co-Frobenius, the bilinear form  $D : H \times H \rightarrow k$ ,  $D(x, y) = t(xS(y))$ , is non-degenerate and  $H^*$ -balanced (see [20, Theorem 3] or [9, Theorem 2]). We have that

$$\begin{aligned} D(x, y) &= t(xS(y)) \\ &= tS(xS(y)) \\ &= t(S^2(y)S(x)) \\ &= D(S^2(y), x) \end{aligned}$$

so the Nakayama automorphism associated to  $D$  is  $S^2$ . By Proposition 6.2 we see that there exists  $u \in U(H^*)$  such that  $S^2(h) = u^{-1} \cdot h \cdot u$  for any  $h \in H$ .

For the converse, assume that  $H$  is unimodular and there exists an invertible  $u \in H^*$  with  $S^2(h) = u^{-1} \cdot h \cdot u$  for any  $h \in H$ . Then let  $t$  be a non-zero left and right integral,

and let  $D : H \times H \rightarrow k$  be the bilinear form associated to  $t$ . Then as above the Nakayama automorphism associated to  $D$  is  $S^2$ , and by using the converse part of Proposition 6.2, we obtain that  $H$  is symmetric as a coalgebra. Note that by using the proof of Proposition 6.2, we see that a symmetric non-degenerate  $H^*$ -balanced bilinear form is

$$B : H \times H \rightarrow k, \quad B(x, y) = D(u^{-1} \cdot x, y) = t((u^{-1} \cdot x)S(y))$$

■

**Remarks 7.2** (i) *The condition that there exists an invertible element  $u \in H^*$  such that  $S^2(h) = u^{-1} \cdot h \cdot u$  for any  $h \in H$  is equivalent to the fact that the map  $(S^2)^*$  is an inner automorphism of the algebra  $H^*$ .*

(ii) *If  $H$  is a Hopf algebra that is symmetric as a coalgebra, then  $S^2$  is not necessarily a inner automorphism of  $H$ . Indeed, let  $H = k[SL_q(2)] = \mathcal{O}_q(SL_2(k))$ , the coordinate ring of  $SL_q(2)$ , with  $q$  not a root of 1 and the characteristic of  $k$  different from 2. It is known that  $H$  is a cosemisimple Hopf algebra. Then  $H$  is symmetric, but as it is proved in [4, Proposition 1.2], no power of the antipode is a inner automorphism.*

(iii) *As a consequence of Theorem 4.2 we obtain many examples of infinite dimensional coalgebras that are co-Frobenius but not symmetric. Indeed, we can take Hopf algebras with non-zero integrals that are not unimodular. A large class of examples like this was constructed in [3] by taking repeated Ore extensions of group algebras, and then factoring by certain Hopf ideals.*

As a consequence we obtain the characterization of finite dimensional Hopf algebras that are symmetric as algebras. This was initially proved in [23]. Other proofs have been given in [21], [10]; see also [14]. Note that the result in [10] is based on a formula concerning the Nakayama automorphism of a unimodular Hopf algebra, which is superseded by a general formula for the modular function on a Hopf algebra or augmented Frobenius algebra in [16]. We recall that for a finite dimensional Hopf algebra  $H$ , a left (resp. right) integral in  $H$  is an element  $t \in H$  such that  $ht = \varepsilon(h)t$  (resp.  $th = \varepsilon(h)t$ ) for any  $h \in H$ . We say that  $H$  is unimodular in the finite dimensional sense if the spaces of left and right integrals in  $H$  are equal.

**Corollary 7.3** *Let  $H$  be a finite dimensional Hopf algebra. Then  $H$  is symmetric as an algebra if and only if  $H$  is unimodular in the finite dimensional sense and  $S^2$  is inner.*

**Corollary 7.4** *Let  $H$  be a cosemisimple Hopf algebra. Then there exists an invertible  $u \in H^*$  such that  $S^2(h) = u^{-1} \cdot h \cdot u$  for any  $h \in H$ .*

The following result was proved in [19] in the particular case of cosemisimple Hopf algebras by using character theory for Hopf algebras.

**Corollary 7.5** *Let  $H$  be a Hopf algebra which is symmetric as a coalgebra. Then  $S^2(A) \subseteq A$  for any subcoalgebra  $A$  of  $H$ .*

**Proof:** Let  $u \in U(H^*)$  such that  $S^2(x) = u^{-1} \cdot x \cdot u$  for any  $x \in H$ . Since  $A$  is a subcoalgebra of  $H$ , it is also an  $(H^*, H^*)$ -sub-bimodule of  $H$ , showing that  $S^2(A) \subseteq A$ . ■

We prove now a general result for semiperfect coalgebras. Recall that if  $X, Y$  are two subspaces of a coalgebra  $C$  with comultiplication  $\Delta$ , then the wedge  $X \wedge Y$  is defined by  $X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y)$ . Also  $\wedge^1 X = X$ , and  $\wedge^n X = (\wedge^{n-1} X) \wedge X$  for  $n \geq 2$ .

**Proposition 7.6** *Let  $C$  be a right semiperfect coalgebra and  $A$  a finite dimensional subcoalgebra of  $C$ . Then  $A_\infty = \bigcup_{n \geq 1} \wedge^n A$  is a finite dimensional subcoalgebra of  $C$ .*

**Proof:** Let us denote by  $\mathcal{C}_A = \{M \in \mathcal{M}^C \mid \rho_M(M) \subseteq M \otimes A\}$ , where  $\rho_M$  denotes the comodule structure map of  $M$ . It is known that  $\mathcal{C}_A$  is a closed subcategory of  $\mathcal{M}^C$  (see [8, Theorem 2.5.5]). Since  $A$  has finite dimension,  $\mathcal{C}_A$  has finitely many types of simple objects. On the other hand,  $\mathcal{C}_{A_\infty}$  is the smallest localizing subcategory of  $\mathcal{M}^C$  that contains  $\mathcal{C}_A$ , so it has the same simple objects as  $\mathcal{C}_A$ . In fact  $M \in \mathcal{C}_{A_\infty}$  if and only if for any  $M' \subseteq M$ ,  $M' \neq M$ , the object  $M/M'$  contains a simple object of  $\mathcal{C}_A$ . Since  $C$  is right semiperfect, we have that  $A_\infty$  is also right semiperfect. But  $A_\infty$  has finitely many types of simple right comodules, so it is finite dimensional by [4, Theorem 2.1]. ■

As an immediate consequence we obtain the following result that was proved in [26] with different methods.

**Corollary 7.7** *Let  $H$  be a Hopf algebra with non-zero integrals and let  $A$  be a finite dimensional subcoalgebra. Then  $A_\infty$  has finite dimension.*

**Corollary 7.8** *Let  $H$  be a Hopf algebra with non-zero integrals. Then  $H_\infty = (k \cdot 1)_\infty$  is a Hopf subalgebra of finite dimension.*

**Corollary 7.9** *Let  $H$  be a Hopf algebra which is symmetric as a coalgebra. Let  $K$  be a Hopf subalgebra such that either  $G(K) = \{1\}$  (i.e.  $K$  is irreducible) or  $G(K) = G(H)$ . Then  $K$  is a symmetric coalgebra.*

**Proof:** It is known that  $K$  has non-zero integrals (however the integral on  $K$  is not necessarily the restriction of the integral on  $H$  to  $K$ , see [4]), and in each of the two situations  $K$  is also unimodular (see [8, Exercise 5.5.10]). Let  $u \in U(H^*)$  as in Theorem 7.1. If  $i : K \rightarrow H$  is the inclusion map, then  $v = i^*(u) \in U(K^*)$ . Since the antipode  $S_K$  of  $K$  is the restriction

of the antipode  $S$  of  $H$ , we have that  $S_K^2(x) = u^{-1} \cdot x \cdot u = v^{-1} \cdot x \cdot v$  for any  $x \in K$ , so  $K$  is symmetric.  $\blacksquare$

**Corollary 7.10** *Let  $H$  be a Hopf algebra which is symmetric as a coalgebra. Then the finite dimensional Hopf algebra  $H_\infty$  from Corollary 7.8 is a symmetric coalgebra.*

**Corollary 7.11** *Let  $H$  be a Hopf algebra which is symmetric as a coalgebra. If  $K$  is a Hopf subalgebra of  $H$  containing the coradical of  $H$ , then  $K$  is also symmetric as a coalgebra.*

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